Spherical Tiling, Spherical Perspectives, and Eversions: Examples of Science and Art Connections

Reza Sarhangi, Towson University <rsarhangi@towson.edu>

__Reza Sarhangi__ is a professor of mathematics and a mathematics educator at Towson University, Maryland. He is the author of a textbook in geometry. He is the director and the proceedings editor of the international conference of Bridges: Mathematical Connections in Art, Music, and Science (www.BridgesMathArt.Org).

Introduction

Almost all human beings are fluent in recognizing and appreciating patterns, in their physical and mathematical senses, and are able to deal effortlessly with the abstractions of language, music, visual art, and theatre. Yet most people think that they have a latent aversion to physics and mathematics and are largely unaware of how deeply embedded these disciplines are in the world around them. Still, we have seen over and over again how fascinated and excited people become when mathematical and physical connections are presented in ways that relate to their experiences and trigger their natural curiosities and aesthetic sensibilities. In this article, we will explore the sphere as a source of inspiration for both scientific and artistic communities via a few examples. After all, we are living on a sphere-like object called Earth and enjoy the beauty of our globe by other sphere-like objects called eyes.

Regular Tiling of the Sphere

Platonic solids were known to humans much earlier than the time of Plato. Theatetus (415 - 369 BC), the Greek mathematician, has been credited for developing a general theory of regular polyhedra and adding the octahedron and icosahedron to solids that were known earlier. The name *Platonic solids* for regular polyhedra comes from the Greek philosopher Plato (427 - 347 BC), who associated them with the "elements" and the cosmos in his book *Timaeus*. The investigation of mathematical orders and relationships in solids fascinated Plato immensely, and he tried to apply this order to other objects of importance in his philosophy: "Elements". Elements, in ancient beliefs, were the four objects that constructed the physical world; these elements are fire, air, earth, and water. Plato suggested that the geometric forms of the smallest particles of these elements are regular polyhedra. Fire is represented by the tetrahedron, earth the cube, air the octahedron, water the icosahedron, and the almost-spherical dodecahedron, the universe.

The Platonic solids and their associations with the elements and the cosmos have been presented in Johannes Kepler's book, *Harmonices Mundi*, as in the following figures (Figure 1):

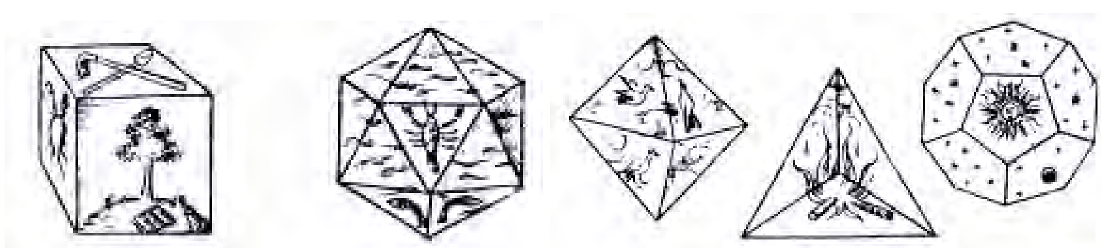

Figure 1

A medieval Persian mathematician, Buzjani, tried to apply the idea of a regular tessellation of a Euclidean plane to a sphere. In Euclidean geometry of the plane, if p indicates the number of sides of a regular polygon and q the number of copies of the regular polygon about each vertex point, then it is easy to show that $(p - 2)(q - 2) = 4$. Therefore, the number of regular tessellations on the Euclidean plane is three; equilateral triangle, {3, 6}, square, {4, 4}, and regular hexagon, {6, 3}.

It is interesting to study the regular tessellation of a sphere. Since the sum of the angle measures of a spherical triangle is more than 180° (Figure 2), the tessellation of a regular p-gon with q copies about each vertex is $(p - 2)(q - 2) < 4$. An important fact about the above formula is the assumption of $p > 2$ for the Euclidean case. However, on the sphere, we may construct regular polygons of only two sides: biangles or lunes (Figure 3). If the angle measure of a biangle is $360°/q$, then we are able to tessellate a sphere with q copies of the biangle.

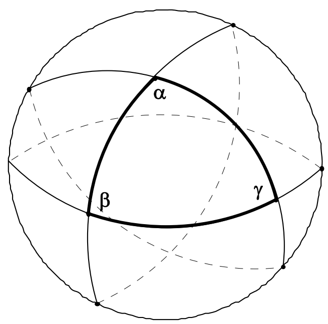

Figure 2

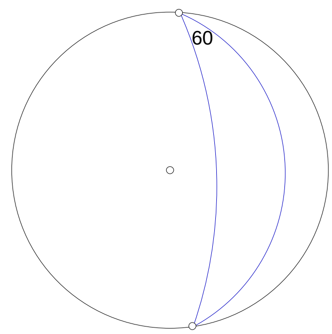

Figure 3

Suppose that for a certain integer, $p > 2$, we are able to construct a spherical regular p-gon tiling with angle measure of $360°/q$. This means a finite number of regular spherical p-gons can cover the sphere without any gaps or overlaps. Now if all the vertices stay in their places on the sphere but the sphere becomes flattened on its p-gon faces, then the resulting structure becomes a Platonic solid! This shows that the ideas of the regular tiling of a sphere for $p > 2$ and Platonic solids are the same. But the challenge of constructing such a regular tiling on a sphere using a compass and straightedge still remains. Before continuing any further, we should make it clear that the “straightedge” tool for such a construction is a great circle and, therefore, line segments on the sphere are arcs of great circles. However, the “compass” is the same tool as we use in Euclidean geometry. Buzjani presents his method of constructing regular tilings on a sphere in an old treatise called *On Those Parts of Geometry Needed by Craftsmen.*

Abul Wafa Buzjani, born in 940 A.D. and raised in Buzjan, a city in Persia, learned mathematics from his uncles and later in his twenties, moved to Baghdad. There, he flourished as a mathematician and astronomer and, in fact, it is believed that he is the person who established trigonometry, as we study it today. Buzjani participated in meetings among artists, artisans, and mathematicians who were concerned with performing and constructing mathematical designs [1].

Here, is illustrated one of Buzjani’s geometric constructions of regular tessellations on a sphere, which he demonstrated in his treatise: the Platonic solid of the dodecahedron. To construct the spherical dodecahedron, consider a sphere S with a given diameter r. First construct segment AB as a diameter on a Euclidean plane and divide it into three congruent segments of AC, CD, and DB. With center D and radius AD we draw a half circle that meets the perpendicular line to AB passing through B at point E. We find H on AB in such a way that $BH = 1/2\ BE$. With the center H and radius HE we find point L on ray AB. BL is a side of a spherical pentagon that covers the sphere (Figure 4).

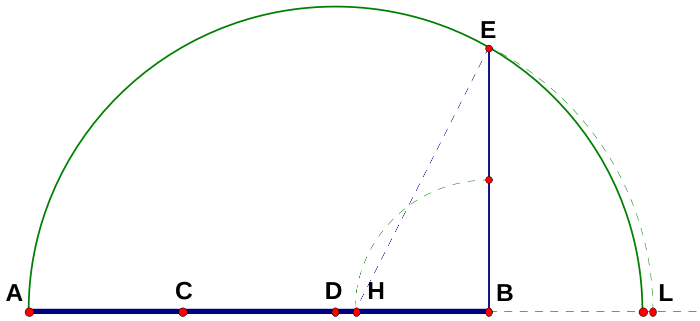

Figure 4

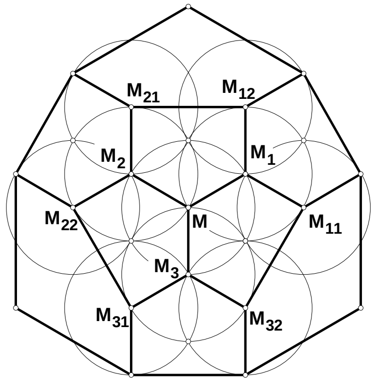

Figure 5

Now choose a point on the sphere, call it M, and draw a circle with radius BL. Divide the circumference of this circle into three congruent arcs of M_1M_2, M_2M_3, and M_3M_1. From each M_i, i =1, 2, 3, we make three circles with radius M_iM and divide each circle into three congruent arcs of MM_{i1}, $M_{i1}M_{i2}$, and $M_{i2}M$. Now we have three regular spherical pentagons of $MM_1M_{12}M_{21}M_2$, $MM_2M_{22}M_{31}M_3$, and $MM_1M_{11}M_{32}M_3$. We continue the process by choosing M_i, i =1, 2, 3, as our new selected point and repeat the construction process that was presented for M to illustrate new spherical pentagons. Continuing the process will result in the regular spherical dodecahedron (Figure 5).

The artistic fascinations that evolved from such geometric constructions on a sphere, which included other spherical Platonic as well as Archimedean solids (solids with two or more different regular faces that have identical vertices), can be observed in the immensely rich and detailed designs of the medieval Persian dome interiors, as can be observed in the following examples (Figure 6).

Figure 6

A Modern Approach to Sphere

Today, artists who are fascinated with the sphere try to find other ways of expressing their artistic explorations. The Northport Public Library in New York commissioned George W. Hart, an artist and computer scientist, to create a unique *Millennium Bookball* sculpture for its' newly expanded Laurel Avenue building. The work is a spherical assemblage of wooden "books," five feet in diameter, hanging in the two-story catalog area of the library. The books are made of various hard woods, with the titles and authors carved in gold leaf. The sculpture was assembled at a community assembly event, something like a barn-raising, but for art (Figure 7).

Figure 7

Spherical Geometric Techniques for Classroom

Magnus Wenninger, a mathematics teacher who was inspired by H.S.M. Coxeter in studying and creating polyhedral models, based on the work of Buckminster Fuller and his models of spherical polyhedra, has created a series of geometric construction techniques that are presented in his book, *Spherical Models*. In this book he not only shows how to create paper models of regular and semiregular spherical models but also how to construct Geodesic domes and other types of models. As an example, let us consider an octahedron and its dual, the cube, as is illustrated in Figure 8. (Note: The dual of a polyhedron is the polyhedron obtained by connecting the centers of the faces.) Projecting their edges onto the surface of the octahedron circumscribing sphere, and including the great circles of the edges of the cube on the sphere, generates forty-eight right spherical triangles. Exactly the same sphere with this set of right spherical triangles can be constructed, if we reverse the process by choosing a cube and its dual octahedron. This is due to the fact that the basic symmetry groups for the spherical octahedron and the spherical cube are the same. Figures 9 and 10 illustrate the layout and final result for creating circular bands for making these right spherical triangles, which have been performed based on a cube with edge length of $e = 2$ that is inscribed in a sphere with radius measure ${}_0R = \sqrt{3}$.

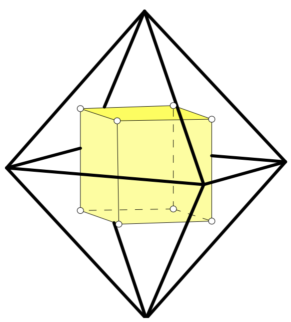

Figure 8

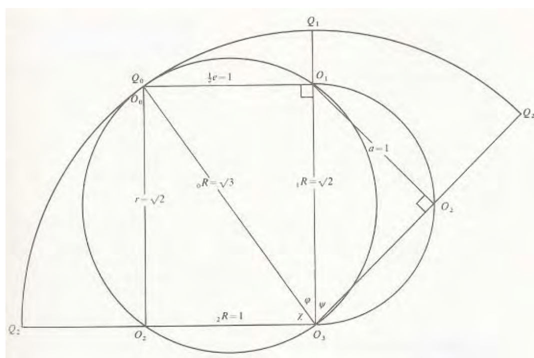

Figure 9

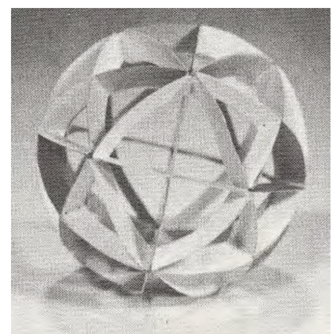

Figure 10

The Six-Point Perspective of Dick Terms

The one-point perspective was discovered by Italian artists during the Renaissance. The two- and three-point perspectives are regular practices for the artist who brings three-dimensionality into play. Dick Terms, an artist who brings the surrounding world around a sphere to its surface, presents his way of presenting a four-point perspective as a cylinder with four continuous vanishing points that have been labeled as the four directions of North, South, East, and West (Figure 11).

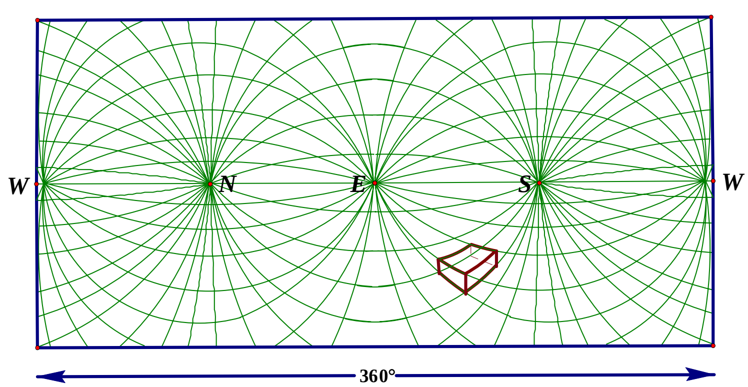

Figure 11

Now consider your eyeball hanging in the middle of a room and you see the objects from all around your eye. In addition to the mentioned four points, you need two more vanishing points, Zenith (above), and Nadir (below), to illustrate your surrounding world on, not a cylinder, but a sphere. This is what Dick

Terms presents as a part of his spherical paintings. He says “Whereas Picasso and the cubists painted a violin as if they were looking all around it; I paint as if I were the violin looking out at the world.” The following figures illustrate some of the Dick Terms’ spheres.

Figure 12

Art and Science Connections in Modern Time

The history of science and art connections witnesses numerous relationships that developed or deepened during centuries among scientists and artists. Brent Collins, an artist, and Carlo Séquin, a computer science professor at the University of California, Berkeley, are among them. Brent’s artwork, which is intuitive, touches science areas such as optimization of energy, topology of minimal surfaces, and symmetries. Carlo’s studying of Brent’s work resulted in discovering scientific relationships and constraints of the work and then generalizing them in order to create other possibilities of the artwork using the computer. In return, Brent can study these virtual sculptures using the computer and then create work based on new findings that is almost impossible to “imagine”. Then his new artwork would be the new area of research for Carlo’s future computer imaginations and productions. The first figure below, Figure 13, is the image of *Music of the Spheres* by Brent Collins, and the second image, Figure 14, is the transition of a conceptual CAD model into a wood sculpture by Carlo Séquin. One is a hand carving by Brent and the other is an image created with a computer by Carlo. What Brent has conceived through his artistic intuition, Carlo has presented using physics and mathematics.

Figure 13

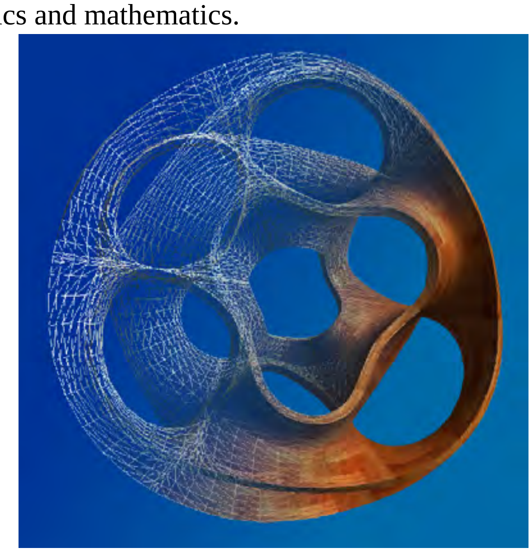

Figure 14

Let alone the public, some mathematical ideas, past a certain level, would be difficult to comprehend by scholars in other branches of science or even mathematicians in other fields. However, to support mathematics, mathematicians need to reach the public and other disciplines. Despite the fact that the Geometry Center at the University of Minnesota, Minneapolis, is closed, some of its productions are still reaching the public. A good example in this regard is the computer-graphics video *Outside In* [2]. In the late 1950s, Steven Smale proved that it is possible to turn a sphere outside-in, which can pass through itself, by means of a continuous deformation, without any puncturing, ripping, creasing, or pinching [3]. A new proof of Smale's theorem that provided more geometric insight was presented by Bill Thurston [4]. The video *Outside In* is based on Thurston's outside-in (eversion).

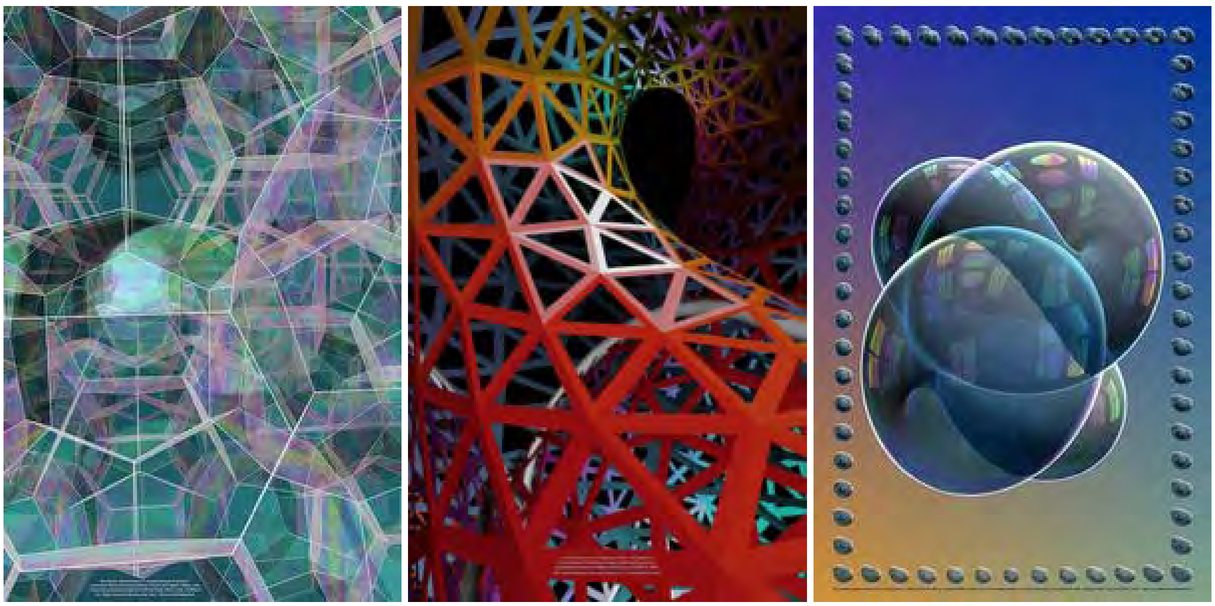

Figure 15

In 1998, John M. Sullivan, working with George Francis, Stuart Levy, and Camille Goudeseune, produced a computer graphics video, *The Optiverse*, which illustrates his new way of eversions that are different from the earlier ones mentioned above in that they are computed by an energy minimization process [5]. The above images are from *The Optiverse* videotape. This video, which presents these minimax sphere eversions with colorful images and interesting music background, not only makes better sense of a challenging mathematics and physics problem, but also brings a sense of appreciation to the public and curiosity to the younger generations – future mathematicians and scientists. Such work illustrates how art can promote mathematics and science in a meaningful and dynamic way.

Conclusion: Some Spherical Geometry Classroom Connections

To make the article more meaningful to the teachers and teacher educators, we would like to invite the readers to visit an online resource that is now available: *Modules for Non-Euclidean Geometries*. For this, you would need to visit the following site that is created by the author of this article:

http://www.towson.edu/~gsarhang/geometry.html

Here, in this resource, the reader may visit the page: *Modules for Non-Euclidean Geometries*. There are three files that are located in that page: (a) *A Proposal for the Introduction of Non-Euclidean Geometry into the Secondary School Geometry Curriculum*, (b) *A Module for Spherical Geometry*, and (c) *A Module for Hyperbolic Geometry.* The proposal and the modules are the outcomes of an independent graduate study course that Anileen Gray, a high school teacher, completed under the supervision of the author of this article. In (a) we read:

> The authors of this article believe that the inclusion of a unit dealing with non-Euclidean geometries into a high school geometry curriculum along the lines of the modules as presented below, has a threefold purpose: Firstly, students are given an opportunity to review much of what they have learnt about Euclidean geometry. Secondly, they learn about two non-Euclidean geometries – spherical and hyperbolic geometries. Finally, they are given the opportunity to compare the postulates and theorems on the plane with the equivalent statements for the sphere and hyperbolic plane. Comparison of the similarities and differences among the three geometries will give students the chance to gain a deeper understanding of geometry as a whole.

In (b) *A Module for Spherical Geometry*, the readers are acquainted with classroom activities for spherical geometry and all the necessary background and tools. We read there:

> It would be worthwhile as an introductory activity to give students a globe of the earth and ask them to find the shortest path for an airplane flying from Washington to Moscow. Encourage students to place a piece of ribbon on the globe joining the two points and convince themselves that the shortest path is in fact an arc of a great circle passing through these two cities. At this stage students should gain an appreciation for the fact that the study of spherical geometry has direct application in the field of aviation. In addition, spherical geometry has applications in many other fields of study such as physics, chemistry, and art.

What follows is a series of student-centered activities in which students are actively involved in discovering similarities and differences between Euclidean geometry and spherical geometry. The approximate time for each activity is shown in parentheses next to the *Note to the Teacher* following each activity. Students should be encouraged to do each activity as it arises and answer the accompanying questions. The teacher may wish to take time out at the end of each period to discuss the students' observations and get feedback from the students on what they have discovered.

References

[1] A. Özdural, *Mathematics and Arts: Connections between Theory and Practice in the Medieval Islamic World*, Historia Mathematica 27, Academic Press, 2000, pp. 171-301.

[2] S. Levy, D. Maxwell, and T. Munzner, *Outside In*, Narrated videotape (21 min) from the Geometry Center, AK Peters, Wellesley, MA, 1994.

[3] S. Smale, *A Classification of Immersions of the Two-Sphere*, Trans. Amer. Math. Soc. P. 90, 1959.

[4] S. Levy, *Making Waves: A Guide to the Ideas Behind "Outside In"*, AK Peters, Wellesley, MA, 1995.

[5] J. M. Sullivan, *"The Optiverse" and Other Sphere Eversions*, Conference Proceedings, Bridges: Art, Music, and Science, pp. 1-20, 1999.

Acknowledgment: The author is grateful to the reviewers' comments and suggestions. The paper has improved immensely because of them and because of a thorough reading and commenting by my dear friend and colleague Professor Michael Krach.